

Bernstein's Lethargy Theorem in Fréchet Spaces

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Abstract. In this paper we consider Bernstein's Lethargy Theorem (BLT) in the context of Fréchet spaces. Let X be an infinite-dimensional Fréchet space and let $\mathcal{V} = \{V_n\}$ be a nested sequence of subspaces of X such that $\overline{V_n} \subseteq V_{n+1}$ for any $n \in \mathbb{N}$ and $X = \overline{\bigcup_{n=1}^{\infty} V_n}$. Let e_n be a decreasing sequence of positive numbers tending to 0. Under an additional natural condition on $\sup\{\text{dist}(x, V_n)\}$, we prove that there exists $x \in X$ and $n_o \in \mathbb{N}$ such that

$$\frac{e_n}{3} \leq \text{dist}(x, V_n) \leq 3e_n$$

for any $n \geq n_o$. By using the above theorem, we prove both Shapiro's [19] and Tyuremskikh's [22] theorems for Fréchet spaces. Considering rapidly decreasing sequences, other versions of the BLT theorem in Fréchet spaces will be discussed. We also give a theorem improving Konyagin's [9] result for Banach spaces.

1 Introduction

Let $C[0, 1]$ denote the space of real valued continuous functions on $[0, 1]$ with the supremum norm $\|\cdot\|$. For $f \in C[0, 1]$, the sequence of best approximations (or equivalently the distance from f to the linear subspace of polynomials P_n of degree $\leq n$), are defined as:

$$\text{dist}(f, P_n) = \rho(f, P_n) = \rho_n(f) = \inf\{\|f - p\| : p \in P_n\}.$$

Clearly $\rho(f, P_1) \geq \rho(f, P_2) \geq \dots$ and $\{\rho(f, P_n)\}$ form a nonincreasing sequence of best approximations. Furthermore we have,

- a) $\rho_n(\lambda f) = |\lambda| \rho_n(f)$
- b) $\rho_n(f + v) = \rho_n(f)$ for $f \in C[0, 1]$ and $v \in P_n$
- c) $\rho_n(f_1 + f_2) \leq \rho_n(f_1) + \rho_n(f_2)$ and $|\rho_n(f_1) - \rho_n(f_2)| \leq \|f_1 - f_2\|$ for $f_1, f_2 \in C[0, 1]$.

The last property implies the continuity of the mapping from $C[0, 1]$ to \mathbb{R}^+ defined as $f \rightarrow \rho_n(f)$.

The density of polynomials in $C[0, 1]$ implies that

$$\lim_{n \rightarrow \infty} \rho_n(f) = 0.$$

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However, the Weierstrass approximation theorem gives no information about the speed of convergence for $\rho_n(f)$. Bernstein, considered “the inverse problem of the theory of best approximation” and showed that for each nonincreasing, null sequence (d_n) , there exist a function $f \in C[0, 1]$ with $\rho_n(f) = d_n$, see [6]. This theorem is sometimes referred as Bernstein’s Lethargy Theorem (BLT) and it has been applied to the theory of quasi analytic functions in several complex variables [15] and used in the constructive theory of functions [20]. Following the proof of Bernstein, Timan [21] extended his result to an arbitrary system of strictly embedded *finite dimensional* subspaces Y_n . Later Shapiro, [19], replacing $C[0, 1]$ with an arbitrary Banach space $(X, \|\cdot\|)$ and the sequence of n -dimensional subspaces of polynomials of degree $\leq n$ by a sequence (Y_n) where $Y_1 \subset Y_2 \subset \dots$ are strictly embedded *closed subspaces* of X , showed that in this setting, for each null sequence (d_n) of non-negative numbers, there is a vector $x \in X$ such that

$$\rho_n(x) = \inf\{\|x - u\| : u \in Y_n\} \neq O(d_n).$$

Thus there is no $M > 0$ such that $\rho_n(x) \leq Md_n$. In other words $\rho_n(x)$ decay arbitrarily slowly. This result was strengthened by Tyuremskikh [22] who established that the sequence of best approximations may converge to zero at an arbitrary slow rate: for any expanding sequence $\{Y_n\}$ of subspaces and for any sequence $\{d_n\}$ of positive numbers converging to zero, he constructed an element $x \in X$ such that $\lim_{n \rightarrow \infty} \rho(x, Y_n) = 0$ and $\rho(x, Y_n) \geq n$. However, it is also possible that the sequence of best approximations may converge to zero arbitrarily fast. For example in [5], under same conditions imposed on $\{Y_n\}$ and $\{d_n\}$, for any sequence $\{c_n\}$ of positive numbers converging to zero, there exists an element $x \in X$ such that $\lim_{n \rightarrow \infty} \frac{\rho(x, Y_n)}{d_n} = 0$ but $\frac{\rho(x, Y_n)}{d_n} \neq O(c_n)$. For a generalization of Shapiro’s theorem we refer the reader to [4] and an application of Tyuremskikh’s theorem to convergence of sequence of bounded linear operators consult [8]. For other versions of Bernstein’s Lethargy Theorem see [1], [2], [5], [13], [17].

These ideas bring us to the following question: given a Banach space X , a strictly increasing sequence $\{Y_n\}$ of subspaces of X with a dense union, and a non-increasing null sequence $\{d_n\} \subset [0, \infty)$, does there exist $x \in X$ such that $\text{dist}(x, Y_n) = d_n$ for each n ? The only known spaces X in which the answer is always “yes” are the Hilbert spaces, [23]. For a general (separable) Banach space X , a solution x is known to exist whenever all Y_n are finite-dimensional [21]. Moreover, it is known that if X has the above property, then it is reflexive, [23]. Later, progress on this question was made by Borodin [7]. Borodin establishes the existence of such an element in case of rapidly decreasing sequences:

Theorem 1.1 (Borodin). *Let X be an arbitrary infinite-dimensional Banach space, $Y_1 \subset Y_2 \subset \dots$ be an arbitrary countable system of strictly nested subspaces in X , and fix a numerical sequence $\{e_n\}_{n=1}^{\infty}$ such that $e_n > \sum_{k=n+1}^{\infty} e_k$ for all natural $n \geq n_0$ with $e_n > 0$. Then there is an element $x \in X$ such that*

$$\rho(x, Y_n) = e_n,$$

for $n = 1, 2, \dots$

Konyagin [9] improved the result of Borodin by proving the following theorem:

Theorem 1.2 (Konyagin). *Let X be a real Banach space, $Y_1 \subset Y_2 \subset \dots$ a sequence of strictly embedded closed linear subspaces of X , and $d_1 \geq d_2 \geq \dots$ a nonincreasing sequence converging to zero, then there exist an element $x \in X$ such that the distance $\rho(x, Y_n)$ from x to Y_n satisfies the inequalities*

$$d_n \leq \rho(x, Y_n) \leq 8d_n \text{ for } n = 1, 2, \dots$$

The aim of this paper is to generalize Theorem 1.2 (see Theorem 2.9) to the case of Fréchet spaces. Since in general approximation theory for Fréchet spaces is quite different from approximation theory in Banach spaces, it is natural to ask whether or not Bernstein's Lethargy Theorem is true in this case. There are several papers concerning Bernstein Lethargy Theorem in the context of Fréchet spaces; we refer the reader to [3], [16], [12] and [1] for progresses in this direction. Notice that a version of Bernstein's lethargy theorem for metrizable topological linear spaces was proved in [12] and Theorem 4.1 in [12] has been applied to the theory of quasianalytic functions in modular function spaces. We refer the reader to [10], [11] for results in this direction.

The main result of this paper is Theorem 2.9. By using Theorem 2.9 we are able to prove both Shapiro's and Tyuremskikh's theorems for Fréchet spaces (see Theorem 2.11 and Theorem 2.12). Theorems 2.5, 2.6 and 2.14 are other versions of the BLT theorem for Fréchet spaces. We also give a theorem improving Konyagin's result for Banach spaces (see Theorem 2.18).

Definition 1.3. $(X, \|\cdot\|)$ is called a *Fréchet space*, if it is a metric linear space which is complete with respect to its F-norm $\|\cdot\|_F$ giving the topology. As usual by an F-norm we mean that $\|\cdot\|_F$ satisfies the following conditions, [18]:

- a) $\|x\|_F = 0$ if and only if $x = 0$,
- b) $\|\alpha x\|_F = \|\alpha\| \|x\|_F$ for all real or complex α with $\|\alpha\| = 1$,
- c) $\|x + y\|_F \leq \|x\|_F + \|y\|_F$,

Many Fréchet spaces X can also be constructed using a countable family of semi norms $\|x\|_k$ where X is a complete space with respect to this family of semi-norms. For example a translation invariant complete metric inducing the topology on X can be defined as

$$d(x, y) = \sum_{k=0}^{\infty} 2^{-k} \frac{\|x - y\|_k}{1 + \|x - y\|_k} \text{ for } x, y \in X.$$

Clearly, every Banach space is a Fréchet space, and the other well known example of a Fréchet space is the vector space $C^\infty[0, 1]$ of all infinitely differentiable functions $f : [0, 1] \rightarrow \mathbb{R}$ where the semi norms are $\|f\|_k = \sup\{|f^k(x)| : x \in [0, 1]\}$. For more information about Fréchet spaces the reader is referred to [18].

2 Results

We start with three technical lemmas.

Lemma 2.1. *Let $(X, \|\cdot\|)$ be a Fréchet space, and let G and V be nonempty subsets of X . Assume that G is a compact set. Then for any $\delta > 0$ there exists a finite set $\{v_1, \dots, v_n\} \subset V$ such that for any $g \in G$*

$$\|g - v_j\| \leq \text{dist}(g, V) + \delta$$

for some $j \in \{1, \dots, n\}$.

Proof. Fix $\delta > 0$ and $g \in G$. Choose $v_g \in V$ such that $\|g - v_g\| < \text{dist}(g, V) + \delta$. Since the function $f_g : G \rightarrow \mathbb{R}$ defined by

$$f_g(h) = \text{dist}(h, V) + \delta - \|h - v_g\|$$

is continuous, there exists an open neighborhood of g , $U_g \subset X$ such that $f_g(h) > 0$ for any $h \in U_g$. Notice that $G \subset \bigcup_{g \in G} U_g$. By the compactness of G , $G \subset \bigcup_{i=1}^n U_{g_i}$. Hence for any $h \in G$

$$\|h - v_{g_i}\| < \text{dist}(g, V) + \delta$$

for some $i \in \{1, \dots, n\}$, which completes the proof. \square

Lemma 2.2. *Fix $\epsilon > 0$. Let (e_n) be a sequence of nonnegative real numbers such that for any $n \in \mathbb{N}$*

$$e_n \geq (2 + \epsilon)e_{n+1}.$$

Set for $n \in \mathbb{N}$ $b_n = \sum_{j=n}^{\infty} 2^{j-n}e_j$. Then $\sum_{n=1}^{\infty} b_n < +\infty$.

Proof. Notice that by d’Alambert’s criterion $\sum_{n=1}^{\infty} 2^n e_n < \infty$. Hence for any $n \in \mathbb{N}$, $b_n \in \mathbb{R}$. Observe that

$$\frac{b_{n+1}}{b_n} = \frac{\lim_m \sum_{j=n+1}^{m+1} 2^{j-n-1}e_j}{\lim_m \sum_{j=n}^m 2^{j-n}e_j}$$

and for $m \geq n$,

$$\frac{\sum_{j=n+1}^{m+1} 2^{j-n-1}e_j}{\sum_{j=n}^m 2^{j-n}e_j} \leq \frac{\sum_{j=n}^m 2^{j-n}e_j}{(2 + \epsilon) \sum_{j=n}^m 2^{j-n}e_j} = \frac{1}{2 + \epsilon}$$

which completes the proof. \square

Lemma 2.3. *Let $\{e_n\}$ be a sequence of positive numbers such that $e_n \geq 3e_{n+1}$ for any $n \in \mathbb{N}$. Then for any $m, n \in \mathbb{N} \setminus \{0\}$,*

$$e_n \geq \sum_{j=n+1}^{n+m} 2^{j-n-1}e_j + 2^m e_{n+m}$$

Proof. It is enough to show this lemma for $n = 1$. We proceed by the induction argument with respect to m . If $m = 1$, then

$$e_1 \geq 3e_2 = e_2 + 2e_2,$$

which shows our claim in this case. Now assume that

$$e_1 \geq \sum_{j=2}^{1+m} 2^{j-2} e_j + 2^m e_{1+m}.$$

Observe that

$$\begin{aligned} e_1 &\geq \sum_{j=2}^{1+m} 2^{j-2} e_j + 2^m e_{1+m} \geq \sum_{j=2}^{1+m} 2^{j-2} e_j + 3(2^m e_{2+m}) \\ &= \sum_{j=2}^{1+m} 2^{j-2} e_j + 2^m e_{2+m} + 2^{m+1} e_{2+m} = \sum_{j=2}^{1+(m+1)} 2^{j-2} e_j + 2^{m+1} e_{1+(m+1)}, \end{aligned}$$

as required. The proof is complete. \square

Now let X be an infinite dimensional Fréchet space equipped with an F-norm $\|\cdot\|$. Assume that $\mathcal{V} = \{V_n\}$ is a nested sequence of linear subspaces of X satisfying $\overline{V_n} \subseteq V_{n+1}$, where the closure is taken with respect to $\|\cdot\|$. Let for $n \in \mathbb{N}$

$$F_n = \{v \in V_{n+1} : \text{dist}(v, V_j) = e_j \text{ for } j = 1, \dots, n\}. \quad (2.1)$$

The first step in obtaining our version of Bernstein's Lethargy Theorem is to show that under some assumptions for any $n \in \mathbb{N}$, F_n is not only nonempty (by the argument of compactness it is sufficient to prove Bernstein's Lethargy Theorem in finite-dimensional setting) but it also consists of elements of a special shape. This will be done in the next lemma. Before presenting it, we introduce some notation. Set for $x \in X$

$$\rho_n(x) = \text{dist}(x, V_n) \quad (2.2)$$

and define

$$d_{n,\mathcal{V}} = \sup\{\rho_n(v) : v \in V_{n+1}\}. \quad (2.3)$$

Let $\{e_n\}$ be a decreasing sequence of positive real numbers satisfying

$$\sum_{j=n}^{\infty} 2^{j-n} e_j < \min\{d_{n,\mathcal{V}}, e_{n-1}\}$$

for any $n \in \mathbb{N}$. (We put $e_0 = \infty$ and $\sum_{j=1}^{\infty} 2^{j-1} e_j < \infty$, if $d_{1,\mathcal{V}} = \infty$.) Fix a sequence (δ_n) of positive numbers such that for any $n \in \mathbb{N}$

$$\sum_{j=n}^{\infty} 2^{j-n} (e_j + \delta_j) < \min\{e_{n-1}, d_{n,\mathcal{V}}\} \quad (2.4)$$

and $v_n \in V_{n+1} \setminus V_n$ such that

$$\sum_{j=n}^{\infty} 2^{j-n}(e_j + \delta_j) = \rho_n(v_n) \geq \|v_n\| - \delta_n. \quad (2.5)$$

Since the function $t \rightarrow \rho_n(tx)$ is continuous for any $x \in X$, $\rho_n(x) = \rho_n(x - v)$ for any $x \in X$ and $v \in V_n$, such v_n exist by 2.4 and the definition of infimum. Then we can state:

Lemma 2.4. *Let $(X, \|\cdot\|)$ be an infinite-dimensional Fréchet space. Let $\{V_n\} = \mathcal{V}$ be a nested sequence of linear subspaces of X satisfying $\overline{V_n} \subseteq V_{n+1}$, where the closure is taken with respect to $\|\cdot\|$. Let $\{e_n\}$ be a decreasing sequence of positive real numbers satisfying 2.4 with a fixed sequence of positive numbers $\{\delta_n\}$. Then for any $n \in \mathbb{N}$ there exists a finite set $Z_n \subset V_n$ and $w_n \in F_n$, such that*

$$w_n = \sum_{j=1}^n q_{j,n},$$

where $q_{j,n} = t_{j,n}v_j - z_{j,n}$, $t_{j,n} \in [0, 1]$, $z_{j,n} \in Z_j$ for $j = 1, \dots, n$ and v_j are given by 2.5. Moreover,

$$\|q_{j,n}\| < \sum_{l=j}^{\infty} 2^{l-j}(e_l + \delta_l).$$

Proof. Notice that for any $n \in \mathbb{N}$ the set $G_n = \{tv_n : t \in [0, 1]\}$ is compact. By Lemma 2.1, applied to G_n and V_n , we can find a finite set $Z_n \subset V_n$ such that for any $g \in G_n$

$$\|g - z_n\| < \rho_n(g) + \delta_n \quad (2.6)$$

for some $z_n \in Z_n$. Fix $n \in \mathbb{N}$. For $k \in \{0, \dots, n-1\}$. Let

$$E_{k,n} = \{v \in V_{n+1} : \rho_j(v) = e_j \text{ for } j = n, \dots, n-k\}.$$

Now we show that there exists $q_{n,n}, \dots, q_{1,n}$, such that $q_{j,n} = t_{j,n}v_j - z_{j,n}$, where $t_{j,n} \in [0, 1]$ and $z_{j,n} \in Z_j$ for $j = 1, \dots, n$ with

$$\sum_{l=j}^n \|q_{l,n}\| < \sum_{l=j}^n 2^{l-j}(e_l + \delta_l) \quad (2.7)$$

for $j = 1, \dots, n$ satisfying

$$w_{k,n} = \sum_{j=n-k}^n q_{j,n} \in E_{k,n} \quad (2.8)$$

for any $k \in \{0, \dots, n-1\}$. First we construct $q_{n,n}$. Notice that the function $f_n(t) = \rho_n(tv_n)$ is continuous and by our assumptions on v_n ,

$$f_n(1) = \sum_{j=n}^{\infty} 2^{j-n}(e_j + \delta_j) > e_n \text{ and } f_n(0) = 0.$$

By the Intermediate Value Theorem there exists $t_{n,n} \in [0, 1]$ such that

$$\rho_n(t_{n,n}v_n) = e_n. \quad (2.9)$$

Again by the continuity of f_n we can assume that $t_{n,n}$ is the smallest number satisfying 2.9. By 2.6 we can choose $z_n \in Z_n$ such that $\|t_{n,n}v_n - z_n\| < \rho_n(t_{n,n}v_n) + \delta_n$. Let

$$q_{n,n} = t_{n,n}v_n - z_n. \quad (2.10)$$

Observe that $\rho_n(q_{n,n}) = e_n$, since $z_n \in V_n$ and V_n is a linear subspace. Hence

$$w_{o,n} = q_{n,n} \in E_{o,n}.$$

Also $q_{n,n}$ satisfies 2.7. Now fix $k \in \{0, \dots, n-2\}$. Assume that we have constructed $q_{n,n}, \dots, q_{n-k,n}$ satisfying 2.8 and 2.7. Now we construct $q_{n-k-1,n}$. By 2.4 applied to $n-k-1$, we can choose $s_{n-k-1} \in [0, 1]$ (the smallest one) such that

$$\rho_{n-k-1}(s_{n-k-1}v_{n-k-1}) = e_{n-k-1} + \sum_{l=n-k}^n 2^{l-n+k}(e_l + \delta_l).$$

Let $u_k = \sum_{l=n-k}^n q_{l,n}$. Observe that

$$\begin{aligned} e_{n-k-1} + \sum_{l=n-k}^n 2^{l-n+k}(e_l + \delta_l) &= \rho_{n-k-1}(s_{n-k-1}v_{n-k-1}) \\ &\leq \rho_{n-k-1}(s_{n-k-1}v_{n-k-1} + u_k) + \rho_{n-k-1}(u_k) \\ &\leq \rho_{n-k-1}(s_{n-k-1}v_{n-k-1} + u_k) + \|u_k\| \\ &\leq \rho_{n-k-1}(s_{n-k-1}v_{n-k-1} + u_k) + \sum_{l=n-k}^n 2^{l-n+k}(e_l + \delta_l). \end{aligned}$$

Hence

$$\rho_{n-k-1}(s_{n-k-1}v_{n-k-1} + u_k) \geq e_{n-k-1}.$$

Let $f_k(t) = \rho_{n-k-1}(tv_{n-k-1} + u_k)$. Observe that

$$f_k(s_{n-k-1}) = \rho_{n-k-1}(s_{n-k-1}v_{n-k-1} + u_k) \geq e_{n-k-1}.$$

Also by our assumptions

$$\begin{aligned} f_k(0) &= \rho_{n-k-1}(u_k) \leq \|u_k\| \leq \sum_{j=n-k}^n \|q_{j,n}\| \\ &< \sum_{l=n-k}^n 2^{l-n+k}(e_l + \delta_l) < \sum_{l=n-k}^{\infty} 2^{l-n+k}(e_l + \delta_l) < e_{n-k-1}. \end{aligned}$$

Since the function f_k is continuous, by the Intermediate Value Theorem there exists $t_{n-k-1} \in [0, s_{n-k-1}]$ such that

$$\rho_{n-k-1}(t_{n-k-1}v_{n-k-1} + u_k) = e_{n-k-1}.$$

By 2.6 we can choose $z_{n-k-1} \in Z_{n-k-1}$ such that

$$\|t_{n-k-1}v_{n-k-1} - z_{n-k-1}\| < \rho_{n-k-1}(t_{n-k-1}v_{n-k-1}) + \delta_{n-k-1}.$$

Let

$$q_{n-k-1,n} = t_{n-k-1}v_{n-k-1} - z_{n-k-1}. \quad (2.11)$$

Since $q_{n-k-1,n} \in V_{n-k}$ and $z_{n-k-1} \in Z_{n-k-1} \subset V_{n-k-1}$, by the above construction

$$w_{k-1,n} = \sum_{j=n-k-1}^n q_{j,n} \in E_{k+1,n},$$

which shows that the equation 2.8 is satisfied. Moreover by the choice of s_{n-k-1} ,

$$\begin{aligned} \|q_{n-k-1,n}\| &= \|t_{n-k-1,n}v_{n-k-1} - z_{n-k-1}\| \\ &< \rho_{n-k-1}(t_{n-k-1,n}v_{n-k-1}) + \delta_{n-k-1} \leq \rho_{n-k-1}(s_{n-k-1}v_{n-k-1}) + \delta_{n-k-1} \\ &= \delta_{n-k-1} + e_{n-k-1} + \sum_{l=n-k}^n 2^{l-n+k}(e_l + \delta_l). \end{aligned}$$

Observe that

$$\begin{aligned} \sum_{l=n-k-1}^n \|q_{l,n}\| &= \|q_{n-k-1,n}\| + \sum_{l=n-k}^n \|q_{l,n}\| \\ &\leq \delta_{n-k-1} + e_{n-k-1} + \sum_{l=n-k}^n 2^{l-n+k}(e_l + \delta_l) + \sum_{l=n-k}^n 2^{l-n+k}(e_l + \delta_l) \\ &= \sum_{l=n-k-1}^n 2^{l-n+k+1}(e_l + \delta_l), \end{aligned}$$

which shows that 2.7 is satisfied. Hence we have constructed $q_{1,n}, \dots, q_{n,n}$ satisfying the requirements of our lemma. In particular, the above reasoning shows that $E_{n-1,n}$ is nonempty and $w_{n-1,n} = \sum_{j=1}^n q_{j,n} \in E_{n-1,n}$. Since $F_n = E_{n-1,n}$, the proof is complete. \square

Applying Lemma 2.4 we now show a version of Bernstein's Lethargy Theorem in Fréchet spaces for rapidly decreasing sequences $\{e_n\}$.

Theorem 2.5. *Let $X, \mathcal{V}, \{d_{n,\mathcal{V}}\}$ and $\{e_n\}$ be such as in Lemma 2.4. Then, there exists $x \in X$ such that $\text{dist}(x, V_n) = e_n$ for any $n \in \mathbb{N}$.*

Proof. Let for any $n \in \mathbb{N}$

$$F_n = \{v \in V_{n+1} : \rho_j(v) = e_j \text{ for } j = 1, \dots, n\}.$$

Take $w_n = \sum_{j=1}^n q_{j,n} \in F_n$ constructed in Lemma 2.4. Fix $j \in \mathbb{N}$. Observe that for any $n \in \mathbb{N}$ $q_{j,n} = t_{j,n}v_j - z_{j,n}$, where $t_{j,n} \in [0, 1]$ and $z_{j,n} \in Z_j$ (see 2.6). Since Z_j is a finite set, we can select a subsequence $\{n_k\}$ and $q_j \in V_{j+1}$ such that

$$\|q_{j,n_k} - q_j\| \rightarrow 0.$$

Applying a diagonal argument we can choose a subsequence $\{n_k\}$ such that for any $j \in \mathbb{N}$,

$$\|q_{j,n_k} - q_j\| \rightarrow 0.$$

Let $s_k = \sum_{j=1}^k q_j$. We show that $\{s_k\}$ is a Cauchy sequence. Fix $\epsilon > 0$. By 2.4 we can find $l \in \mathbb{N}$ such that

$$\sum_{j=l}^{\infty} 2^{j-l} e_j < \epsilon.$$

Notice that by 2.4 and Lemma 2.4 for $n > k \geq l + 1$,

$$\begin{aligned} \|s_k - s_n\| &= \left\| \sum_{j=k+1}^n q_j \right\| \leq \sum_{j=k+1}^n \|q_j\| \\ &\leq \sum_{j=k+1}^n \left(\sum_{m=j}^{\infty} 2^{m-j} (e_m + \delta_m) \right) \leq \sum_{j=k}^{n-1} e_j \leq \epsilon, \end{aligned}$$

which affirms our claim. Since X is complete, $\|s_k - x\| \rightarrow 0$, where

$$x = \sum_{j=1}^{\infty} q_j.$$

Now we show that $\|w_{n_k} - x\| \rightarrow 0$. To do that, fix $\epsilon > 0$ and $n_o \in \mathbb{N}$ such that

$$\sum_{j=n_o}^{\infty} 2^{j-n_o} e_j < \epsilon.$$

Note that for 2.4 and Lemma 2.4, $n_k \geq n_o + 1$, and

$$\begin{aligned} \|w_{n_k} - x\| &\leq \left\| \sum_{j=1}^{n_o} q_{j,n_k} - q_j \right\| + \left\| \sum_{j=n_o+1}^{n_k} q_{j,n_k} - \sum_{j=n_o+1}^{\infty} q_j \right\| \\ &\leq \sum_{j=1}^{n_o} \|q_{j,n_k} - q_j\| + \sum_{j=n_o+1}^{\infty} \|q_{j,n_k}\| + \sum_{j=n_o+1}^{\infty} \|q_j\| \\ &\leq \sum_{j=1}^{n_o} \|q_{j,n_k} - q_j\| + 2 \left(\sum_{j=n_o+1}^{\infty} \left(\sum_{l=j}^{\infty} 2^{l-j} (e_l + \delta_l) \right) \right) \leq \left\| \sum_{j=1}^{n_o} q_{j,n_k} - q_j \right\| + 2 \sum_{j=n_o}^{\infty} e_j \end{aligned}$$

$$\leq \sum_{j=1}^{n_o} \|q_{j,n_k} - q_j\| + 2\epsilon.$$

Since $\|q_{j,n_k} - q_j\| \rightarrow_k 0$ for any $j \in \mathbb{N}$,

$$\sum_{j=1}^{n_o} \|q_{j,n_k} - q_j\| + 2\epsilon \leq 3\epsilon$$

for k sufficiently large, which shows that $\|w_{n_k} - x\| \rightarrow 0$.

Now we show that $\rho_j(x) = e_j$ for $j \in \mathbb{N}$. Fix $j_o \in \mathbb{N}$. Then for $k \geq k_o$ $n_k \geq j_o$. Since $w_{n_k} \in F_{n_k}$,

$$\rho_{j_o}(w_{n_k}) = e_{j_o} \text{ for } k \geq k_o.$$

Hence

$$\rho_{j_o}(x) = \lim_k \rho_{j_o}(w_{n_k}) = e_{j_o},$$

which completes the proof. \square

Theorem 2.6. *Let X , \mathcal{V} , and $\{d_{n,\mathcal{V}}\}$ be such as in Lemma 2.4. Assume that $\{e_n\}$ is a decreasing sequence of nonnegative numbers satisfying $e_n < d_{n,\mathcal{V}}$ and $e_n \geq 3e_{n+1}$ for any $n \in \mathbb{N}$. Then there exists $x \in X$ such that $\text{dist}(x, V_n) = e_n$ for any $n \in \mathbb{N}$.*

Proof. To prove our result, it is sufficient to verify if the assumptions of Theorem 2.5 are satisfied. But this is a consequence of Lemma 2.2 and Lemma 2.3. \square

Now we show that Theorem 2.6 can be applied to prove a version of Bernstein's Lethargy Theorem in the case of any decreasing sequence $\{e_n\}$ of positive numbers tending to 0 and any nested sequence $\{V_n\}$ of linear subspaces of a Fréchet space X , which is the main result of this paper. To do that, being inspired by [9], we need the following construction of a decreasing sequence $\{f_k\}$ of positive numbers tending to zero and a subsequence $\{n_k\}$. Put $f_1 = e_1$ and $n_1 = 1$. If $e_1 \geq 3e_2$, then we define $n_2 = 2$ and $f_2 = e_2$. If $e_1 < 3e_2$ then $f_2 = \frac{f_1}{3}$ and

$$n_2 = \max\{n \geq 2 : f_1 \leq 3e_n\}.$$

Since $f_1 = e_1 > 0$ and $e_n \rightarrow 0$, n_2 is well-defined. Now assume that we have constructed positive numbers f_1, \dots, f_k and $n_1, \dots, n_k \in \mathbb{N}$. We will construct f_{k+1} and n_{k+1} . If

$$e_{n_k+1} \leq \frac{f_k}{3}$$

then we define $f_{k+1} = e_{n_k+1}$ and $n_{k+1} = n_k + 1$. In the opposite case $f_{k+1} = \frac{f_k}{3}$ and

$$n_{k+1} = \max\{n \geq n_k + 1 : f_k \leq 3e_n\}.$$

Since $f_k > 0$ and $e_n \rightarrow 0$, n_{k+1} is well-defined.

Example 2.7. In the above discussion, if we take $e_n = \frac{1}{n}$, then $f_k = \frac{1}{3^{k-1}}$ and $n_k = 3^{k-1}$.

Lemma 2.8. Let $\{e_n\}$ be a decreasing sequence of positive numbers tending to 0. Let $\{f_k\}$ and $\{n_k\}$ be as in the above construction. Then for any $k \in \mathbb{N}$, $f_k \geq 3f_{k+1}$ and $e_{n_k+1} \leq f_k$.

Proof. Fix $k \in \mathbb{N}$. If $f_{k+1} = e_{n_k+1}$, then by our construction

$$f_{k+1} = e_{n_k+1} \leq \frac{f_k}{3}.$$

In the opposite case $f_{k+1} = \frac{f_k}{3}$, which validates our claim.

Analogously, if $f_{k+1} = e_{n_k+1}$, then

$$e_{n_k+1} \leq \frac{f_k}{3} < f_k.$$

If $f_{k+1} \neq e_{n_k+1}$ and $f_k \neq e_{n_{k-1}+1}$, then by definition of n_k , $3e_{n_k+1} \leq f_{k-1} = 3f_k$. Finally, if $f_{k+1} \neq e_{n_k+1}$ and $f_k = e_{n_{k-1}+1}$, then

$$e_{n_k+1} \leq e_{n_{k-1}+1} = f_k,$$

since the sequence $\{e_n\}$ is decreasing. □

Now we are ready to state the main result of this paper.

Theorem 2.9. Let X be a infinite-dimensional Fréchet space and let $\mathcal{V} = \{V_n\}$ be a nested sequence of subspaces of X such that $\overline{V_n} \subseteq V_{n+1}$ for any $n \in \mathbb{N}$ and

$$X = \overline{\bigcup_{n=1}^{\infty} V_n}.$$

Let e_n be a decreasing sequence of positive numbers tending to 0. Assume that

$$d_{\mathcal{V}} = \inf\{d_{n,\mathcal{V}} : n \in \mathbb{N}\} > 0. \quad (2.12)$$

Then there exists $n_o \in \mathbb{N}$ and $x \in X$ such that for any $n \geq n_o$

$$\frac{e_n}{3} \leq \text{dist}(x, V_n) \leq 3e_n.$$

Proof. Let $\{f_k\}$ and $\{n_k\}$ be two sequences associated with $\{e_n\}$ by our construction. Set for $k \in \mathbb{N}$, $W_k = V_{n_k}$, and $\mathcal{V}_1 = \{W_k\}$. Since for any $k \in \mathbb{N}$, $V_{n_k+1} \subset W_{k+1}$,

$$d_{k,\mathcal{V}_1} \geq d_{n_k,\mathcal{V}} \geq d_{\mathcal{V}} > 0.$$

By Lemma 2.8 $f_k \geq 3f_{k+1}$ for any $k \in \mathbb{N}$. Fix $k_o \in \mathbb{N}$ such that for $k \geq k_o$

$$f_k < d_{\mathcal{V}} \leq d_{k,\mathcal{V}_1}.$$

Applying Theorem 2.6 to $\{f_k : k \geq k_o\}$ and $\{W_k : k \geq k_o\}$, there exists $x \in X$ such that

$$\text{dist}(x, V_{n_k}) = f_k \text{ for } k \geq k_o.$$

Let $n_o = n_{k_o} + 1$. Fix $n \geq n_o$. Then there exists exactly one $k \in \mathbb{N}$ such that $n_k < n \leq n_{k+1}$. If $f_{k+1} = e_{n_k+1}$, then by our construction $n_{k+1} = n_k + 1$ and

$$\frac{e_n}{3} < e_n = e_{n_k+1} = f_{k+1} = \text{dist}(x, V_n) < 3e_n.$$

If $f_{k+1} \neq e_{n_k+1}$, then by Lemma 2.8 and our assumptions

$$\text{dist}(x, V_n) \geq \text{dist}(x, V_{n_{k+1}}) = f_{k+1} = \frac{f_k}{3} \geq \frac{e_{n_k+1}}{3} \geq \frac{e_n}{3}.$$

Also if $f_{k+1} \neq e_{n_k+1}$, then, by definition of n_{k+1} ,

$$\text{dist}(x, V_n) \leq \text{dist}(x, V_{n_k}) = f_k \leq 3e_n.$$

The proof is complete. \square

Remark 2.10. If $d_V = 0$ the above result holds true with the same proof for sequences $\{e_n\}$ satisfying $e_n < d_{n,V}$ for $n \geq n_o$.

From Theorem 2.9 we can easily obtain a version of Shapiro's theorem, [19], and a version of Tyuremskikh's theorem, [22], for Fréchet spaces.

Theorem 2.11. Let the assumptions of Theorem 2.9 be satisfied. Then there exists $x \in X$ such that $\rho_n(x) \neq O(e_n)$.

Proof. By Theorem 2.9 applied to the sequence $\{\sqrt{e_n}\}$ there exists $x \in X$ and $n_o \in \mathbb{N}$ such that

$$\frac{\sqrt{e_n}}{3} \leq \rho_n(x) \leq 3\sqrt{e_n}$$

for $n \geq n_o$. Since $e_n \rightarrow 0$, it is obvious that $\rho_n(x) \neq O(e_n)$. \square

Theorem 2.12. Let the assumptions of Theorem 2.9 be satisfied. Then there exists $x \in X$ and $n_o \in \mathbb{N}$ such that $\rho_n(x) \geq e_n$ for $n \geq n_o$.

Proof. By Theorem 2.9 applied to the sequence $\{3\sqrt{e_n}\}$ there exists $x \in X$ and $n_o \in \mathbb{N}$ such that

$$\sqrt{e_n} \leq \rho_n(x) \leq 9\sqrt{e_n}.$$

for $n \geq n_o$. Since $e_n \rightarrow 0$, it is obvious that $\rho_n(x) \geq \sqrt{e_n} \geq e_n$ for $n \geq n_o$. \square

Note that the parameter d_V defined by 2.12 can be equal to 0 which is illustrated in the next example.

Example 2.13. Let X be a space of complex sequences with the Fréchet norm of $x \in X$ given by $\|x\| = \sum_{i=0}^{\infty} 2^{-i} \frac{|x_i|}{1 + |x_i|}$. Set for $n \in \mathbb{N}$, $V_n = \{x \in X : x_j = 0 \text{ for } j \geq n\}$. It is easy to see that

$$d_{n,V} = \sup\{\rho_n(v) : v \in X\} = \frac{1}{2^n}$$

and consequently that $d_V = 0$.

However, if a Fréchet space $(X, \|\cdot\|)$ is equipped with an s -convex F -norm for some $s \in (0, 1]$, which means

$$\|tx\| = |t|^s \|x\| \text{ for any } x \in X, \quad (2.13)$$

then obviously for any linear subspace V of X and $x \in X$, $\text{dist}(tx, V) = |t|^s \text{dist}(x, V)$. In particular this equality holds true in Banach spaces with $s = 1$. Hence we can state the following theorem.

Theorem 2.14. *Let X be a Fréchet space with an s -convex F -norm and let $\{V_n\}$ be a nested sequence of subspaces of X such that $\overline{V_n} \subseteq V_{n+1}$ for any $n \in \mathbb{N}$. Let e_n be a decreasing sequence of positive numbers tending to 0. Then there exists $x \in X$ such that for any $n \in \mathbb{N}$*

$$\frac{e_n}{3} \leq \text{dist}(x, V_n) \leq 3e_n.$$

Proof. It is easy to see that by 2.13 $d_V = \infty$. Hence reasoning as in Theorem 2.9 the proof is complete. \square

In particular we have:

Theorem 2.15. *Let X be a Banach space and let $\{V_n\}$ be a nested sequence of subspaces of X such that $\overline{V_n} \subseteq V_{n+1}$ for any $n \in \mathbb{N}$. Let e_n be a decreasing sequence of positive numbers tending to 0. Then there exists $x \in X$ such that for any $n \in \mathbb{N}$*

$$\frac{e_n}{3} \leq \text{dist}(x, V_n) \leq 3e_n.$$

In the case of Banach spaces, applying Theorem 1.1 instead of Theorem 2.6 we can improve Theorem 1.2. We need the following lemma:

Lemma 2.16. *Let $\{e_n\}$ be a sequence of positive numbers such that $e_n \geq 2e_{n+1}$ for any $n \in \mathbb{N}$. Then for any $n \in \mathbb{N} \setminus \{0\}$,*

$$e_n \geq \sum_{j=n+1}^{\infty} e_j$$

Proof. It is enough to show that for any $n \in \mathbb{N}$, $e_1 \geq \sum_{j=2}^n e_n + e_n$. If $n = 2$, then $e_1 \geq 2e_2 = e_2 + e_2$, which shows our claim for $n = 2$. The general case can be easily obtained by an induction argument with respect to n . \square

Now we slightly modify the construction of a decreasing sequence $\{f_k\}$ of positive numbers tending to zero and a subsequence $\{n_k\}$ given in the case of Frechét spaces. Put $f_1 = e_1$ and $n_1 = 1$. If $e_1 \geq 2e_2$, then we define $n_2 = 2$ and $f_2 = e_2$. If $e_1 < 2e_2$ then $f_2 = \frac{f_1}{2}$ and

$$n_2 = \max\{n \geq 2 : f_1 \leq 2e_n\}.$$

Since $f_1 = e_1 > 0$ and $e_n \rightarrow 0$, n_2 is well-defined. Now assume that we have constructed positive numbers f_1, \dots, f_k and $n_1, \dots, n_k \in \mathbb{N}$. We will construct f_{k+1} and n_{k+1} . If

$$e_{n_k+1} \leq \frac{f_k}{2}$$

then we define $f_{k+1} = e_{n_k+1}$ and $n_{k+1} = n_k + 1$. In the opposite case $f_{k+1} = \frac{f_k}{2}$ and

$$n_{k+1} = \max\{n \geq n_k + 1 : f_k \leq 2e_n\}.$$

Since $f_k > 0$ and $e_n \rightarrow 0$, n_{k+1} is well-defined. Reasoning as in Lemma 2.8 we can prove:

Lemma 2.17. *Let $\{e_n\}$ be a decreasing sequence of positive numbers tending to 0. Let $\{f_k\}$ and $\{n_k\}$ be as in the above construction. Then for any $k \in \mathbb{N}$, $f_k \geq 2f_{k+1}$ and $e_{n_k+1} \leq f_k$.*

Now we can state

Theorem 2.18. *Let X be a Banach space and let $\{V_n\}$ be a nested sequence of subspaces of X such that $\overline{V_n} \subseteq V_{n+1}$ for any $n \in \mathbb{N}$. Let e_n be a decreasing sequence of positive numbers tending to 0. Then there exists $x \in X$ such that for any $n \in \mathbb{N}$*

$$\frac{e_n}{2} \leq \text{dist}(x, V_n) \leq 2e_n.$$

Proof. First we show that Theorem 1.1 remains true under the weaker assumption

$$e_n \geq \sum_{j=n+1}^{\infty} e_j$$

for any $n \in \mathbb{N}$. To prove this modification, for fixed $k \in \mathbb{N}$, and a sequence $\{e_n\}$ satisfying the required condition, set $e_{n,k} = e_n + \frac{1}{k3^n}$. It is clear that for any $n \in \mathbb{N}$,

$$e_{n,k} > \sum_{j=n+1}^{\infty} e_{j,k} \text{ for any } n \in \mathbb{N}.$$

By Theorem 1.1 there exists $x_k \in X$ such that $\rho_n x_k = e_{n,k}$ for any $n \in \mathbb{N}$. Moreover, by the proof of Theorem 1.1 (see [7])

$$x_k = \sum_{n=1}^{\infty} \lambda_{n,k} q_n,$$

where $q_n \in X$, $\|q_n\| \leq 2$ and $|\lambda_{n,k}| \leq e_{n,k} \leq e_{n,1}$ for any $n \in \mathbb{N}$. By the compactness argument and a diagonal process we can assume that there exists a subsequence $\{k_l\}$ such that

$$\lambda_{n,k_l} \rightarrow_l \lambda_n$$

for any $n \in \mathbb{N}$. Notice that for any $n \in \mathbb{N}$ $|\lambda_n| \leq e_{n,1}$. Since the series $\sum_{n=1}^{\infty} e_{n,1}$ is convergent and $\|q_n\| \leq 2$ for any $n \in \mathbb{N}$ the vector

$$x = \sum_{n=1}^{\infty} \lambda_n q_n$$

is well-defined and obviously $\|x_{k_l} - x\| \rightarrow_l 0$. Consequently, by the continuity of the function ρ_n , for any $n \in \mathbb{N}$

$$\rho_n(x_{k_l}) = e_{n,k_l} \rightarrow_l e_n = \rho_n(x),$$

which shows our modification.

Now the proof of Theorem 2.18 proceeds in a similar way as the proof of Theorem 2.9. Instead of Theorem 2.6, Lemma 2.3 and Lemma 2.8, we should apply our modified construction, Theorem 1.1 in our modified version, Lemma 2.16 and Lemma 2.17. \square

From Theorem 2.9 we can easily deduce a version of Bernstein's Lethargy Theorem in which subspaces are replaced by sets W_n satisfying

$$\overline{\text{span}(W_n)} \subseteq W_{n+1}, \text{ for } n \in \mathbb{N}. \quad (2.14)$$

Theorem 2.19. *Let X be a Frechét space and let $\{W_n\}$ be a nested sequence of subsets of X . Set $V_n = \overline{\text{span}(W_n)}$ for $n \in \mathbb{N}$. Let $\{V_n\}$ satisfy 2.14 and the assumptions of Theorem 2.9. Suppose that e_n is a decreasing sequence of positive numbers tending to 0. Then there exists $n_o \in \mathbb{N}$ and $x \in X$ such that for any $n \geq n_o$*

$$\frac{e_n}{3} \leq \text{dist}(x, W_n) \leq 3e_{n-1}.$$

Proof. Notice that by 2.14 for any $x \in X$ and $n \in \mathbb{N}$

$$\text{dist}(x, V_n) \leq \text{dist}(x, W_n) \leq \text{dist}(x, V_{n-1}).$$

By Theorem 2.9 there exists $x \in X$ and $n_o \in \mathbb{N}$ such that

$$\frac{e_n}{3} \leq \text{dist}(x, V_n) \leq 3e_n.$$

for $n \geq n_o$, which shows our claim. \square

Remark 2.20. *If we assume additionally that there exists $M > 0$ such that $\frac{e_n}{e_{n+1}} \leq M$ for any $n \in \mathbb{N}$, then we obtain a stronger version of Theorem 2.19. In this case there exists $n_o \in \mathbb{N}$ and $x \in X$ such that for any $n \geq n_o$*

$$\frac{e_n}{3} \leq \text{dist}(x, W_n) \leq 3Me_n.$$

Notice that the assumption 2.14 in Theorem 2.19 is necessary.

Example 2.21. Let X be a Banach space and let $W_n = \{x \in X : \|x\| \leq n\}$. Let $\{e_n\}$ be a decreasing sequence of positive numbers tending to zero. Since $X = \bigcup_{n=1}^{\infty} W_n$, there is no $x \in X$ such that

$$\frac{e_n}{3} \leq \text{dist}(x, W_n) \leq 3e_{n-1}.$$

At the end of this section we show some results concerning Fréchet spaces $(X, \|\cdot\|)$ and sequences of subspaces of X $\mathcal{V} = \{V_n\}$ satisfying or not satisfying the assumption 2.12. First notice that by [12], Prop. 3.4 and Cor. 3.8, if for all n , V_n are finite-dimensional, and $X = \overline{\bigcup_{n=1}^{\infty} V_n}$ then $d_{\mathcal{V}} > 0$ provided $R(\mathcal{V}) > 0$, where

$$R(\mathcal{V}) = \inf\{\sup\{\|tv\| : t \in \mathbb{R}_+, v \in \left(\bigcup_{n=1}^{\infty} V_n\right) \setminus \{0\}\}\}.$$

In general, we have the following:

Lemma 2.22. *Let $(X, \|\cdot\|)$ be an infinite-dimensional Fréchet space. Let $\mathcal{V} = \{V_n\}$ be a nested sequence of linear subspaces of X satisfying $\overline{V_n} \subseteq V_{n+1}$ such that*

$$X = \overline{\bigcup_{n=1}^{\infty} V_n}.$$

If $d_{\mathcal{V}} > 0$ then $R(\mathcal{V}) > 0$.

Proof. Notice that for any $x \in X$, $n \in \mathcal{N}$ and $t \in \mathcal{R}_+$, $\|tx\| \geq \rho_n(tx)$. Since $X = \overline{\bigcup_{n=1}^{\infty} V_n}$, this implies that

$$R(\mathcal{V}) \geq d_{\mathcal{V}},$$

which proves our claim. \square

Remark 2.23. *By [12], Prop. 3.4 and Cor. 3.8 and Lemma 2.22 if all n , V_n are finite-dimensional, then $d_{\mathcal{V}} > 0$ if and only if $R(\mathcal{V}) > 0$. We do not know if this is satisfied for arbitrary \mathcal{V} .*

Lemma 2.24. *Let $(X, \|\cdot\|)$ be an infinite-dimensional Fréchet space. Let $\mathcal{V} = \{V_n\}$ be a nested sequence of linear subspaces of X satisfying $\overline{V_n} \subseteq V_{n+1}$ such that*

$$X = \overline{\bigcup_{n=1}^{\infty} V_n}.$$

Let $\|\cdot\|_1$ be an F -norm defined on X equivalent to $\|\cdot\|$. Denote

$$\rho_{n,1}(x) = \inf\{\|x - v\|_1 : v \in V_n\},$$

$$d_{n,\mathcal{V},1} = \sup\{\rho_{n,1}(x) : x \in V_{n+1}\}$$

and

$$d_{\mathcal{V},1} = \inf\{d_{n,\mathcal{V},1} : n \in \mathcal{N}\}.$$

Then $d_{\mathcal{V}} = 0$ if and only if $d_{\mathcal{V},1} = 0$.

Proof. Assume that $d_{\mathcal{V}} = 0$ and $d_{\mathcal{V},1} > 0$. Then there exists $\epsilon > 0$ such that $d_{n,\mathcal{V},1} > \epsilon$ for any $n \in \mathbb{N}$. This implies that we can find $x_n \in V_{n+1}$ such that

$$\rho_{n,1}(x_n) \geq \epsilon/2.$$

On the other hand, since $d_{\mathcal{V}} = 0$, $\lim_n \rho_n(x_n) = 0$ and consequently, there exist $v_n \in V_n$ such that $\lim_n \|x_n - v_n\| = 0$. Since $\|\cdot\|_1$ is equivalent to $\|\cdot\|$

$$\epsilon/2 \leq \rho_{n,1}(x_n) \leq \|x_n - v_n\|_1 \rightarrow 0;$$

a contradiction. \square

As an application of Lemma 2.24 and Theorem 2.14 we can state the following theorem.

Theorem 2.25. *Let $(X, \|\cdot\|)$ be a locally bounded Fréchet space and let $\{V_n\}$ be a nested sequence of subspaces of X such that $\overline{V_n} \subseteq V_{n+1}$ for any $n \in \mathbb{N}$. Let e_n be a decreasing sequence of positive numbers tending to 0. Then there exists $x \in X$ and $n_o \in \mathbb{N}$ such that for any $n \geq n_o$*

$$\frac{e_n}{3} \leq \text{dist}(x, V_n) \leq 3e_n.$$

Proof. By [18], p.95, Th. 3.2.1, there exists $0 < p \leq 1$ and a p -homogenous norm equivalent to $\|\cdot\|$. By Lemma 2.24 and Theorem 2.14 we get our result. \square

In particular applying Theorem 2.25 and [18], p. 96, Th. 3.2.2 we get

Theorem 2.26. *Let $(X, \|\cdot\|)$ be a locally convex Fréchet space and let $\{V_n\}$ be a nested sequence of subspaces of X such that $\overline{V_n} \subseteq V_{n+1}$ for any $n \in \mathbb{N}$. Let e_n be a decreasing sequence of positive numbers tending to 0. Then there exists $x \in X$ and $n_o \in \mathbb{N}$ such that for any $n \geq n_o$*

$$\frac{e_n}{3} \leq \text{dist}(x, V_n) \leq 3e_n.$$

Now we state a theorem concerning Fréchet spaces in which the topology is determined by a sequence of p_n -homogenous norms.

Theorem 2.27. *Let X be an infinite dimensional linear space and let $\{\|\cdot\|_n\}$ be a family of p_n -homogenous F -pseudonorms which is total over X , i.e for any $x \in X$, if $\|x\|_n = 0$ for any $n \in \mathbb{N}$, then $x = 0$. Define on X an F -norm $\|\cdot\|$ by*

$$\|x\| = \sum_{j=1}^{\infty} \frac{\|x\|_j}{2^j(1 + \|x\|_j)}.$$

Assume that $(X, \|\cdot\|)$ is a Fréchet space. Let $\mathcal{V} = \{V_n\}$ be a nested sequence of linear subspaces of X satisfying $\overline{V_n} \subseteq V_{n+1}$ such that

$$X = \overline{\bigcup_{n=1}^{\infty} V_n}.$$

Assume that there exists $N \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ we can find $x_n \in V_{n+1}$ such that

$$\text{dist}_j(x_n, V_n) = \inf\{\|x_n - v\|_j : v \in V_n\} > 0$$

for some $j \in \{1, \dots, N\}$. Then $d_{\mathcal{V}} > 0$.

Proof. By our assumptions for any $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$,

$$\rho_n(tx_n) \geq \text{dist}_j(tx_n, V_n) = t^{p_j} \text{dist}_j(x_n, V_n)$$

for some $j \in \{1, \dots, N\}$. Hence $d_{n, \nu} \geq \frac{1}{2^N}$ for any $n \in \mathbb{N}$ which shows that $d_\nu \geq \frac{1}{2^N}$. \square

Example 2.28. Let $\{g_n\} \subset C^\infty[0, 1]$ be a fixed sequence of orthonormal functions with respect to the scalar product $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$. Let

$$V_n = \text{span}[g_1, g_2, \dots, g_{2n-1}, g_{2n}, g_{2n+2}, \dots, g_{2(n+k)} : k \in \mathbb{N}].$$

Let us equip X with the F-norm $\|\cdot\|$ given by

$$\|x\| = \sum_{j=1}^{\infty} \frac{\|x\|_n}{2^n(1 + \|x\|_n)},$$

where $\|x\|_n = \sup\{|x^{(n)}(t)| : t \in [0, 1]\}$. Notice that for any $n \in \mathbb{N}$, and $v \in V_n$,

$$\|g_{2n+1} - v\|_1 \geq \left(\int_0^1 (g_{2n+1} - v)^2 dt\right)^{1/2} \geq \left(\int_0^1 (g_{2n+1})^2 dt\right)^{1/2} = 1.$$

Hence in this case the assumptions of Theorem 2.27 are satisfied with $N = 1$. Consequently, $d_\nu > 0$ and by Theorem 2.9 for any decreasing sequence of positive numbers $\{e_n\}$ tending to 0 there exists $n_o \in \mathbb{N}$ and $x \in X$ such that for any $n \geq n_o$

$$\frac{e_n}{3} \leq \text{dist}(x, V_n) \leq 3e_n.$$

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